Border Collision Bifurcations in n-Dimensional Piecewise Linear Discontinuous Maps

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Abstract.

In this paper we report some important results that help in analizing the border collision bifurcations that occur in n-dimensional discontinuous maps. For this purpose, we use the piecewise linear approximation in the neighborhood of the plane of discontinuity. Earlier, Feigin had made a similar analysis for general n-dimensional piecewise smooth continuous maps. Proceeding along similar lines, we obtain the general conditions of existence of period-1 and period-2 fixed points before and after a border collision bifurcation. The application of the method is then illustrated using a specific example of a two-dimensional discontinuous map.

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1. Introduction

In recent times a significant amount of research effort has been directed towards nonsmooth bifurcations because a large number of physical and engineering systems have been found to exhibit this class of bifurcation phenomena [1, 2, 3]. Such physical systems are generally given by two or more sets of differential equations, and the system switches from one set to another when some switching conditions are satisfied. It has been shown that such systems give rise to piecewise smooth maps, where the discrete-time state space is divided into two or more compartments with different functional forms of the map, separated by borderlines [4, 5]. A new type of bifurcation, called *border collision bifurcation* occurs when a fixed point collides with a borderline, resulting in an abrupt change in the Jacobian matrix. Most past studies on border collision bifurcations have concentrated on piecewise smooth maps that are continuous across the borderlines [6, 7, 8, 9, 10, 11].

However, it has also been found that some switching dynamical systems, for example, the Colpitts oscillator [12], dc-dc converters [13], thyristor controlled reactor circuits [14], sigma-delta modulators [15] and many other electronic circuits [16] give rise to discontinuous maps. Discontinuity in the map arises if, in the n dimensional Poincaré section there exists an $(n-1)$ dimensional manifold such that infinitesimally close points at the two sides of the manifold map to points that are far apart. In order to explain the bifurcation phenomena in such systems, it is necessary to have a bifurcation theory for discontinuous maps.

The bifurcation theory for 1D and 2D continuous piecewise smooth maps has been developed in [8, 9]. In another important line of work, Feigin obtained some results on the existence of periodic orbits in n-dimensional piecewise smooth continuous maps, which were brought to the notice of the English speaking audience in [11]. The theory for discontinuous maps is in the preliminary stage of development, with some progress reported for one-dimensional maps in [17, 18].

In the present paper, we follow Feigin's approach in case of discontinuous maps, and present new results that are applicable to the general n-dimensional case. In particular, we present the conditions of existence of the period-1 and period-2 fixed points which form the basis of the observable bifurcation phenomena in such maps. For this purpose, we use a piecewise linear normal form where the discontinuity is defined by two parameters.

Finally, the application of these results is illustrated by deducing the bifurcation behavior of a two-dimensional discontinuous map, and validating the results by simulation.

2. The Normal Form of the Discontinuous Map

We consider discontinuous maps of the form

$$\bar{\mathbf{X}}^{(k+1)} = \begin{cases} \Pi^-(\bar{\mathbf{X}}^k) & \text{for} \quad \bar{\mathbf{X}} \in P^- \\ \Pi^+(\bar{\mathbf{X}}^k) & \text{for} \quad \bar{\mathbf{X}} \in P^+ \end{cases}$$

where the discrete-time phase space is divided into two regions P^- and P^+, and Π^- and Π^+ represent the functional forms of the map in the two regions respectively.

Since our points of interest are the occurrence of bifurcations with the traversal of a fixed point across the border, we can study the same by applying local linear approximation in neighborhood of the discontinuity:

$$\mathbf{X}^{(k+1)} = F(\mathbf{X}^{(k)}) = \begin{cases} \mathbf{A}_1\mathbf{X}^{(k)} + \mathbf{C}\mu_1, & \text{if } x_1^{(k)} < 0 \\ \mathbf{A}_2\mathbf{X}^{(k)} + \mathbf{C}\mu_2, & \text{if } x_1^{(k)} > 0 \end{cases} \qquad (1)$$

where

$$\mathbf{A}_1 = \left.\frac{\partial\Pi^-}{\partial\mathbf{X}}\right|_{\mathbf{X}=0}, \quad \mathbf{A}_2 = \left.\frac{\partial\Pi^+}{\partial\mathbf{X}}\right|_{\mathbf{X}=0}$$

and

$$\mathbf{C} = \left.\frac{\partial\Pi^-}{\partial\mu_1}\right|_{\mu_1=0} = \left.\frac{\partial\Pi^+}{\partial\mu_2}\right|_{\mu_2=0}.$$

Here $\mathbf{X}^{(k+1)}$ is $(k+1)$th iteration of $n \times 1$ matrix $\mathbf{X} = (x_1, x_2, \ldots, x_n)^T$, $\mathbf{A}_1$ and $\mathbf{A}_2$ are $n \times n$ matrices, $\mathbf{C} = (c_1, c_2, \ldots, c_n)^T$ is $n \times 1$ matrix and μ_1, μ_2 are parameters.

We can apply a suitable coordinate transformation to obtain the matrices in the normal form, where the matrices $\mathbf{A}_1 = [a_{ij}^{(1)}]$ and $\mathbf{A}_2 = [a_{ij}^{(2)}]$ are such that $a_{ij}^{(1)} = a_{ij}^{(2)}$ if $j \neq 1$, i.e., except the first columns of the matrices all other columns are same ‖. In the earlier work on piecewise smooth continuous maps [11, 9] this condition was required to ensure continuity of the map. In the present work, though we are considering discontinuous maps, we retain this property of the matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ and move the discontinuity to the parameters μ_1 and μ_2 without loss of generality. In [17] the length of the discontinuity was considered to be constant. In this paper, in order to allow a variable length of the discontinuity, we have used two parameters μ_1 and μ_2. We have assumed $\mathbf{C}$ to be same on both sides, which physically means that the two parameters have proportional effect on the two sides of the border.

2.1. Existence of period-1 fixed points

Let us consider two fixed points $\mathbf{X}_L^* = (x_{1L}^*, x_{2L}^*, \ldots, x_{nL}^*)^T$ and $\mathbf{X}_R^* = (x_{1R}^*, x_{2R}^*, \ldots, x_{nR}^*)^T$ of the sub-mappings Π^- and Π^+ respectively. That is,

$$\mathbf{X}_L^* = \mathbf{A}_1\mathbf{X}_L^* + \mathbf{C}\mu_1 \quad \text{for} \quad x_{1L}^* < 0, \qquad (2)$$

$$\mathbf{X}_R^* = \mathbf{A}_2\mathbf{X}_R^* + \mathbf{C}\mu_2 \quad \text{for} \quad x_{1R}^* > 0. \qquad (3)$$

‖ The derivation of the normal form in one-dimension can be found in [8], and that in two-dimension can be found in [19, 9]

where x_{1L}^* and x_{1R}^* are the first components of $\mathbf{X}_L^*$ and $\mathbf{X}_R^*$ respectively.

We define $p^*(\lambda) = |\lambda \mathbf{I} - \mathbf{A}_1|$ and $p^{**}(\lambda) = |\lambda \mathbf{I} - \mathbf{A}_2|$, where $\mathbf{I}$ is the $n \times n$ identity matrix ¶.

Assuming $(\mathbf{I} - \mathbf{A}_1)$ and $(\mathbf{I} - \mathbf{A}_2)$ to be invertible, we find

$$\mathbf{X}_L^* = (\mathbf{I} - \mathbf{A}_1)^{-1}\mathbf{C}\mu_1 = \frac{\mathrm{adj}(\mathbf{I}-\mathbf{A}_1)}{p^*(1)}\mathbf{C}\mu_1,$$

$$\mathbf{X}_R^* = (\mathbf{I} - \mathbf{A}_2)^{-1}\mathbf{C}\mu_2 = \frac{\mathrm{adj}(\mathbf{I}-\mathbf{A}_2)}{p^{**}(1)}\mathbf{C}\mu_2.$$

Now the kth element of these matrices can be written as

$$x_{kL}^* = \frac{b_{kL}^*}{p^*(1)}\mu_1, \quad x_{kR}^* = \frac{b_{kR}^*}{p^{**}(1)}\mu_2 \tag{4}$$

where $b_{kL}^* = [(\mathrm{adj}(\mathbf{I} - \mathbf{A}_1))\mathbf{C}]_k$ is the kth element of the column matrix $(\mathrm{adj}(\mathbf{I} - \mathbf{A}_1))\mathbf{C}$ and $b_{kR}^* = [(\mathrm{adj}(\mathbf{I} - \mathbf{A}_2))\mathbf{C}]_k$ is the kth element of the column matrix $(\mathrm{adj}(\mathbf{I} - \mathbf{A}_2))\mathbf{C}$. Now, since in the matrices $\mathbf{A}_1$ and $\mathbf{A}_2$, $a_{ij}^{(1)} = a_{ij}^{(2)}$ if $j \neq 1$, the cofactor of first column of $(\mathbf{I} - \mathbf{A}_1)$ and $(\mathbf{I} - \mathbf{A}_2)$ are same. Hence

$$b_{1L}^* = \sum_{i=1}^{n} b_{1i}^{(1)} c_i = \sum_{i=1}^{n} b_{1i}^{(2)} c_i = b_{1R}^* = b_1^* \quad (\text{say})$$

where $b_{1i}^{(1)}$ and $b_{1i}^{(2)}$, are the elements of the first rows of the matrices $(\mathrm{adj}(\mathbf{I} - \mathbf{A}_1))$ and $(\mathrm{adj}(\mathbf{I} - \mathbf{A}_2))$ respectively and c_i are the elements of the column matrix $\mathbf{C}$. Using the above in (4) we obtain

$$x_{1L}^* = \frac{b_1^*}{p^*(1)}\mu_1, \quad x_{1R}^* = \frac{b_1^*}{p^{**}(1)}\mu_2 \tag{5}$$

We require $x_{1L}^* < 0$ for $\mathbf{X}_L^*$ to exist and $x_{1R}^* > 0$ for $\mathbf{X}_R^*$ to exist. With the help of above derivations we can now attempt a classification of the possible behavior at the border collision bifurcation.

Equation (5) points to two possibilities:

Case 1: $p^*(1)p^{**}(1) > 0$. This will be true if both $p^*(1)$ and $p^{**}(1)$ are positive or if both are negative. There are two sub-cases.

Case 1a: If $\mu_1\mu_2 > 0$ then only one fixed point will exist.

Case 1b: If $\mu_1\mu_2 < 0$ then either two fixed points will exist or no fixed point will exist.

Case 2: $p^*(1)p^{**}(1) < 0$. This will be true if $p^*(1) > 0$ and $p^{**}(1) < 0$, or if $p^*(1) < 0$ and $p^{**}(1) > 0$. Here also there are two sub-cases.

Case 2a: If $\mu_1\mu_2 > 0$ then either two fixed points will exist or there will not be any fixed point.

Case 2b: If $\mu_1\mu_2 < 0$ then only one fixed point will exist.

Table 1 and Table 2 summarize these results for $b_1^* > 0$ and $b_1^* < 0$ respectively. From these tables the bifurcation sequences can be inferred for a given set of parameters, as μ_1 and μ_2 are varied.

¶ Note that Feigin [11] defined these in the opposite way, as $p^*(\lambda) = |\mathbf{A}_1 - \lambda\mathbf{I}|$ and $p^{**}(\lambda) = |\mathbf{A}_2 - \lambda\mathbf{I}|$.

Table 1. The conditions of existence of fixed points when $b_1^* > 0$.

Conditions on the parameters			Existence of fixed points			
$p^*(1)p^{**}(1)$	$p^*(1)$	$p^{**}(1)$	$\mu_1 > 0$ $\mu_2 < 0$	$\mu_1 > 0$ $\mu_2 > 0$	$\mu_1 < 0$ $\mu_2 > 0$	$\mu_1 < 0$ $\mu_2 < 0$
> 0	< 0	< 0	$\mathbf{X}_L^* \ \& \ \mathbf{X}_R^*$	$\mathbf{X}_L^*$	None	$\mathbf{X}_R^*$
< 0	> 0	< 0	$\mathbf{X}_R^*$	None	$\mathbf{X}_L^*$	$\mathbf{X}_L^* \ \& \ \mathbf{X}_R^*$
> 0	> 0	> 0	None	$\mathbf{X}_R^*$	$\mathbf{X}_L^* \ \& \ \mathbf{X}_R^*$	$\mathbf{X}_L^*$
< 0	< 0	> 0	$\mathbf{X}_L^*$	$\mathbf{X}_L^* \ \& \ \mathbf{X}_R^*$	$\mathbf{X}_R^*$	None

Table 2. The conditions of existence of fixed points when $b_1^* < 0$.

Conditions on the parameters			Existence of fixed points			
$p^*(1)p^{**}(1)$	$p^*(1)$	$p^{**}(1)$	$\mu_1 > 0$ $\mu_2 < 0$	$\mu_1 > 0$ $\mu_2 > 0$	$\mu_1 < 0$ $\mu_2 > 0$	$\mu_1 < 0$ $\mu_2 < 0$
> 0	< 0	< 0	None	$\mathbf{X}_R^*$	$\mathbf{X}_L^* \ \& \ \mathbf{X}_R^*$	$\mathbf{X}_L^*$
< 0	> 0	< 0	$\mathbf{X}_L^*$	$\mathbf{X}_L^* \ \& \ \mathbf{X}_R^*$	$\mathbf{X}_R^*$	None
> 0	> 0	> 0	$\mathbf{X}_L^* \ \& \ \mathbf{X}_R^*$	$\mathbf{X}_L^*$	None	$\mathbf{X}_R^*$
< 0	< 0	> 0	$\mathbf{X}_R^*$	None	$\mathbf{X}_L^*$	$\mathbf{X}_L^* \ \& \ \mathbf{X}_R^*$

3. Conditions for the existence of period-2 orbit

Suppose that a new two period solution originates at a border collision point. Let the two points of the period-2 orbit be $\mathbf{M}^* = (m_1^*, m_2^*, \ldots, m_n^*)^T$ and $\mathbf{M}^{**} = (m_1^{**}, m_2^{**}, \ldots, m_n^{**})^T$ lying on the half planes L and R respectively.

    Now we derive the condition for the existence of the period-2 orbit. Using the local map (1), we can write

$$f(\mathbf{M}^*) = \mathbf{M}^{**} \quad \text{and} \quad f(\mathbf{M}^{**}) = \mathbf{M}^*$$

i.e.,

$$\mathbf{M}^* = \mathbf{A}_2 \mathbf{M}^{**} + \mathbf{C}\mu_2, \tag{6}$$

$$\mathbf{M}^{**} = \mathbf{A}_1 \mathbf{M}^* + \mathbf{C}\mu_1. \tag{7}$$

    Now subtracting (6) from (7) we get

$$\Delta\mathbf{M} = \mathbf{A}_1 \mathbf{M}^* - \mathbf{A}_2 \mathbf{M}^{**} + \mathbf{C}(\mu_1 - \mu_2) \tag{8}$$

where $\Delta\mathbf{M} = \mathbf{M}^{**} - \mathbf{M}^* = (m_1^{**} - m_1^*, m_2^{**} - m_2^*, \ldots, m_n^{**} - m_n^*)^T = (\delta m_1, \delta m_2, \ldots, \delta m_n)^T$.

    Now as $a_{ij}^{(1)} = a_{ij}^{(2)} = a_{ij}$ if $j \neq 1$, writing (8) in scalar form we obtain:

$$\delta m_k = a_{k1}^{(1)} m_1^* - a_{k1}^{(2)} m_1^{**} + c_k(\mu_1 - \mu_2) + \sum_{j=2}^{n} a_{kj}(m_j^* - m_j^{**}), \tag{9}$$

for $k = 1, 2, \ldots, n$.

    Equation (9) can be rewritten by adding and subtracting $a_{k1}^{(1)} m_1^{**}$ as

$$\delta m_k = (a_{k1}^{(1)} - a_{k1}^{(2)})m_1^{**} + c_k(\mu_1 - \mu_2) + \sum_{j=1}^{n} a_{kj}^{(1)}(m_j^* - m_j^{**}).$$

Similarly by adding and subtracting $a_{k1}^{(2)}m_1^*$ in equation (9) we get

$$\delta m_k = (a_{k1}^{(1)} - a_{k1}^{(2)})m_1^* + c_k(\mu_1 - \mu_2) + \sum_{j=1}^{n} a_{kj}^{(2)}(m_j^* - m_j^{**}).$$

The above two equations can be written in vector form as

$$\Delta\mathbf{M} + \mathbf{A}_1\Delta\mathbf{M} = [\mathbf{A}_1 - \mathbf{A}_2]_1 \, m_1^{**} + \mathbf{C}(\mu_1 - \mu_2) \tag{10}$$

$$\Delta\mathbf{M} + \mathbf{A}_2\Delta\mathbf{M} = [\mathbf{A}_1 - \mathbf{A}_2]_1 \, m_1^* + \mathbf{C}(\mu_1 - \mu_2) \tag{11}$$

where $[\mathbf{A}_1 - \mathbf{A}_2]_1$ indicate the first column of $[\mathbf{A}_1 - \mathbf{A}_2]$.

Assuming $(\mathbf{I} + \mathbf{A}_2)$ to be invertible, from (11) we obtain:

$$\Delta\mathbf{M} = \frac{\mathbf{B}^*}{p^{**}(-1)}m_1^* + \frac{\mathbf{D}^*}{p^{**}(-1)}(\mu_1 - \mu_2) \tag{12}$$

where $\mathbf{B}^* = (-1)^n(\mathrm{adj}(\mathbf{I} + \mathbf{A}_2))[\mathbf{A}_1 - \mathbf{A}_2]_1$ and $\mathbf{D}^* = (-1)^n(\mathrm{adj}(\mathbf{I} + \mathbf{A}_2))\mathbf{C}$. Similarly assuming $(\mathbf{I} + \mathbf{A}_1)$ to be invertible, from (10) we obtain

$$\Delta\mathbf{M} = \frac{\mathbf{B}^{**}}{p^*(-1)}m_1^{**} + \frac{\mathbf{D}^{**}}{p^*(-1)}(\mu_1 - \mu_2) \tag{13}$$

where $\mathbf{B}^{**} = (-1)^n(\mathrm{adj}(\mathbf{I} + \mathbf{A}_1))[\mathbf{A}_1 - \mathbf{A}_2]_1$ and $\mathbf{D}^{**} = (-1)^n(\mathrm{adj}(\mathbf{I} + \mathbf{A}_1))\mathbf{C}$.

Now writing (12) and (13) in scalar form

$$\delta m_k = \frac{b_k^*}{p^{**}(-1)}m_1^* + \frac{d_k^*}{p^{**}(-1)}(\mu_1 - \mu_2) \tag{14}$$

$$\delta m_k = \frac{b_k^{**}}{p^*(-1)}m_1^{**} + \frac{d_k^{**}}{p^*(-1)}(\mu_1 - \mu_2). \tag{15}$$

Since $a_{kj}^{(1)} = a_{kj}^{(2)} = a_{kj}$ for $j \neq 1$ and $k = 1, 2, \ldots, n$, we have $b_1^{**} = b_1^* = b_1$ and $d_1^{**} = d_1^* = d_1$. Hence from (14) and (15)

$$\frac{b_1}{p^{**}(-1)}m_1^* + \frac{d_1}{p^{**}(-1)}\mu_0 = \frac{b_1}{p^*(-1)}m_1^{**} + \frac{d_1}{p^*(-1)}\mu_0, \tag{16}$$

where $\mu_0 = (\mu_1 - \mu_2)$. From (14)

$$m_k^{**} = m_k^* + \frac{b_k^*}{p^{**}(-1)}m_1^* + \frac{d_k^*}{p^{**}(-1)}\mu_0, \tag{17}$$

substituting (17) into (7), we obtain (in scalar form)

$$m_k^* = \sum_{j=1}^{n} a_{kj}^{(1)} m_j^* - \frac{b_k^*}{p^{**}(-1)}m_1^* - \frac{d_k^*}{p^{**}(-1)}\mu_0 + c_k\mu_1$$

or equivalently using matrices:

$$\mathbf{M}^* = \hat{\mathbf{A}}_1\mathbf{M}^* + \mathbf{C}\mu_1 - \frac{\mathbf{D}^*}{p^{**}(-1)}\mu_0 \tag{18}$$

where

$$\hat{\mathbf{A}}_1 = \mathbf{A}_1 - \frac{1}{p^{**}(-1)} \begin{pmatrix} b_1^* & 0 & \cdots & 0 \\ b_2^* & 0 & \cdots & 0 \\ \cdots\cdots\cdots\cdots \\ b_n^* & 0 & \cdots & 0 \end{pmatrix}. \tag{19}$$

Assuming $(\mathbf{I} - \hat{\mathbf{A}}_1)$ to be invertible, we can now solve $\mathbf{M}^*$. From (18)

$$\mathbf{M}^* = (\mathbf{I} - \hat{\mathbf{A}}_1)^{-1} \left(\mathbf{C}\mu_1 - \frac{\mathbf{D}^*}{p^{**}(-1)}\mu_0 \right). \tag{20}$$

Now

$$|\mathbf{I} - \hat{\mathbf{A}}_1| = \frac{|\mathbf{I}p^{**}(-1) - \mathbf{A}_1 p^{**}(-1) + \mathbf{L}_1|}{p^{**}(-1)} = \frac{d}{p^{**}(-1)}, \tag{21}$$

where

$$\mathbf{L}_1 = \begin{pmatrix} b_1^* & 0 & \cdots & 0 \\ b_2^* & 0 & \cdots & 0 \\ \cdots\cdots\cdots\cdots \\ b_n^* & 0 & \cdots & 0 \end{pmatrix}$$

and d is some constant.

Hence from (20) and (21) we obtain

$$\begin{aligned} \mathbf{M}^* &= \frac{\mathrm{adj}(\mathbf{I} - \hat{\mathbf{A}}_1)p^{**}(-1)}{d} \left(\mathbf{C}\mu_1 - \frac{\mathbf{D}^*}{p^{**}(-1)}\mu_0 \right) \\ &= \hat{\mathbf{K}}_1 p^{**}(-1)\mu_1 - \hat{\mathbf{K}}_2(\mu_1 - \mu_2), \end{aligned} \tag{22}$$

where

$$\hat{\mathbf{K}}_1 = \frac{\mathrm{adj}(\mathbf{I} - \hat{\mathbf{A}}_1)\mathbf{C}}{d} \quad \text{and} \quad \hat{\mathbf{K}}_2 = \frac{\mathrm{adj}(\mathbf{I} - \hat{\mathbf{A}}_1)\mathbf{D}^*}{d}.$$

Writing (22) in scalar form

$$m_1^* = k_1\mu_1 p^{**}(-1) - k_2(\mu_1 - \mu_2), \tag{23}$$

and substituting this into (16) we obtain

$$m_1^{**} = k_1\mu_1 p^*(-1) - k_3(\mu_1 - \mu_2). \tag{24}$$

Hence in two parameter n-dimensional discontinuous map (1), a period-2 orbit will occur only when the following two conditions are simultaneously satisfied:

$$k_1 p^{**}(-1)\mu_1 - k_2(\mu_1 - \mu_2) < 0, \tag{25}$$
$$k_1 p^*(-1)\mu_1 - k_3(\mu_1 - \mu_2) > 0. \tag{26}$$

where k_1 and k_2 are respectively the first components of $\hat{\mathbf{K}}_1$ and $\hat{\mathbf{K}}_2$, and

$$k_3 = k_2 \frac{p^*(-1)}{p^{**}(-1)} + \frac{d_1}{b_1} \left(1 - \frac{p^*(-1)}{p^{**}(-1)} \right). \tag{27}$$

The derivation of the conditions for the existence of orbits of periodicity greater than 2 in the n-dimensional discontinuous map is very cumbersome. This is still an open problem.

Note that (25) and (26) give the conditions of existence of period-2 orbit. In addition, one has to consider the stability of the orbit in order to determine whether the period-2 orbit will actually be observable in the asymptotically stable behavior of the system.

4. Example: A Two Dimensional Piecewise Linear Discontinuous Map in Normal Form

The normal form for the two dimensional discontinuous map is

$$
\begin{pmatrix} x^{(n+1)} \\ y^{(n+1)} \end{pmatrix} = \begin{cases} \mathbf{A}_1 \begin{pmatrix} x^{(n)} \\ y^{(n)} \end{pmatrix} + \mathbf{C}\mu_1, & \text{if } x^{(n)} < 0 \\ \\ \mathbf{A}_2 \begin{pmatrix} x^{(n)} \\ y^{(n)} \end{pmatrix} + \mathbf{C}\mu_2, & \text{if } x^{(n)} > 0 \end{cases} \tag{28}
$$

where

$$
\mathbf{A}_1 = \begin{pmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{pmatrix}, \quad \mathbf{A}_2 = \begin{pmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},
$$

τ_L is the trace and δ_L is the determinant of the Jacobian matrix $\mathbf{A}_1$ of the system at a fixed point in $L := \{(x, y) \in \Re^2 : x < 0\}$ and τ_R is the trace and δ_R is the determinant of the Jacobian matrix $\mathbf{A}_2$ of the system at a fixed point in $R := \{(x, y) \in \Re^2 : x > 0\}$. The normal form (28) follows from that derived for piecewise smooth continuous maps in [6, 10], with discontinuity added through the parameters μ_1 and μ_2.

In case of the map (28), the terms $p^*(1)$, $p^{**}(1)$, $p^*(-1)$ and $p^{**}(-1)$ are given by

$$
p^*(1) \quad = |\mathbf{I} - \mathbf{A}_1| = \begin{vmatrix} 1 - \tau_L & -1 \\ \delta_L & 1 \end{vmatrix} = (1 - \tau_L + \delta_L),
$$

$$
p^{**}(1) \quad = |\mathbf{I} - \mathbf{A}_2| = \begin{vmatrix} 1 - \tau_R & -1 \\ \delta_R & 1 \end{vmatrix} = (1 - \tau_R + \delta_R).
$$

$$
p^*(-1) \quad = |-\mathbf{I} - \mathbf{A}_1| = \begin{vmatrix} -1 - \tau_L & -1 \\ \delta_L & -1 \end{vmatrix} = (1 + \tau_L + \delta_L),
$$

$$
p^{**}(-1) = |-\mathbf{I} - \mathbf{A}_2| = \begin{vmatrix} -1 - \tau_R & -1 \\ \delta_R & -1 \end{vmatrix} = (1 + \tau_R + \delta_R).
$$

4.1. Inference about period-1 fixed points

Now if we choose $p^*(1) > 0$ and $p^{**}(1) > 0$, i.e., $p^*(1)p^{**}(1) > 0$, application of the results of n-dimensional discontinuous map (for $n = 2$) will yield the following inferences.

If we keep μ_2 constant and vary μ_1, then

- if $\mu_1\mu_2 > 0$ then only one period-1 fixed point will exist,
- if $\mu_1\mu_2 < 0$ then either two period-1 fixed points will exist or no period-1 fixed point will exist.

We now show, using the specific 2D map (28) that the above results are true.

4.2. Proof of the existence of $\mathbf{X}_L^*$ and $\mathbf{X}_R^*$ fixed points

If a fixed point $\mathbf{X}_L^* = (x_L^*, \, y_L^*)^T$ exists, it is given by the solution of the system of equations

$$
\begin{pmatrix} x_L^* \\ y_L^* \end{pmatrix} = \begin{pmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{pmatrix} \begin{pmatrix} x_L^* \\ y_L^* \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu_1
$$

$$
= \frac{\mu_1}{1 - \tau_L + \delta_L} \begin{pmatrix} 1 \\ -\delta_L \end{pmatrix}. \tag{29}
$$

Now the fixed point will exist if $x_L^* < 0$, i.e.,

$$
\frac{\mu_1}{1 - \tau_L + \delta_L} = \frac{\mu_1}{p^*(1)} < 0 \tag{30}
$$

This is true if

$$
\{\tau_L > 1 + \delta_L \text{ or equivalently } p^*(1) < 0\} \quad \text{for } \mu_1 > 0, \tag{31}
$$

$$
\{\tau_L < 1 + \delta_L \text{ or equivalently } p^*(1) > 0\} \quad \text{for } \mu_1 < 0. \tag{32}
$$

The above conditions (31) and (32) are depicted graphically in Fig. 1(a).
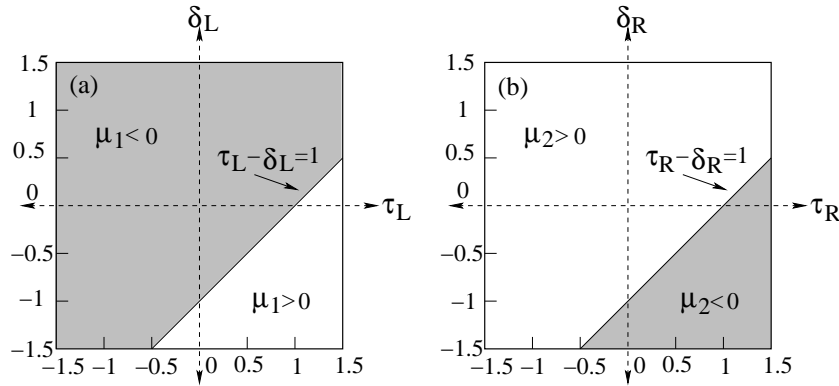
Figure 1. (a) Regions of existence of the $\mathbf{X}_L^*$ fixed point as μ_1 is varied through 0. The fixed point exists in the shaded region if $\mu_1 < 0$, and in the unshaded region if $\mu_1 > 0$. (b) Regions of existence of the $\mathbf{X}_R^*$ fixed point as μ_2 is varied through 0. The fixed point exists in the shaded region if $\mu_2 < 0$, and in the unshaded region if $\mu_2 > 0$.

In a similar manner it can be shown that the fixed point $\mathbf{X}_R^*$ will exist if

$$
\frac{\mu_2}{1 - \tau_R + \delta_R} = \frac{\mu_2}{p^{**}(1)} > 0 \tag{33}
$$

This is true if

$$
\{\tau_R > 1 + \delta_R \text{ or equivalently } p^{**}(1) < 0\} \quad \text{for } \mu_2 < 0, \tag{34}
$$

$$
\{\tau_R < 1 + \delta_R \text{ or equivalently } p^{**}(1) > 0\} \quad \text{for } \mu_2 > 0. \tag{35}
$$

The above conditions (34) and (35) are depicted graphically in Fig. 1(b).

The conditions (31), (32), (34), and (35) divide the (τ_L, τ_R) parameter space into four regions R_1, R_2, R_3 and R_4 as shown in Fig. 2. These regions correspond to the four rows in Table 1.
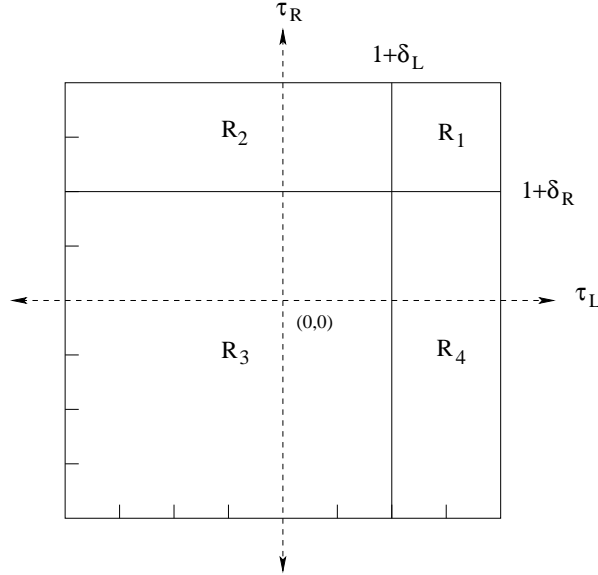

Figure 2. Partitioning of the parameter space where different bifurcations should be observed.

In the region R_1, since $\tau_L > 1 + \delta_L$ and $\tau_R > 1 + \delta_R$, the period one fixed points $\mathbf{X}_L^*$ and $\mathbf{X}_R^*$ are not stable. Therefore, even though they exist, the trajectory diverges to infinity.
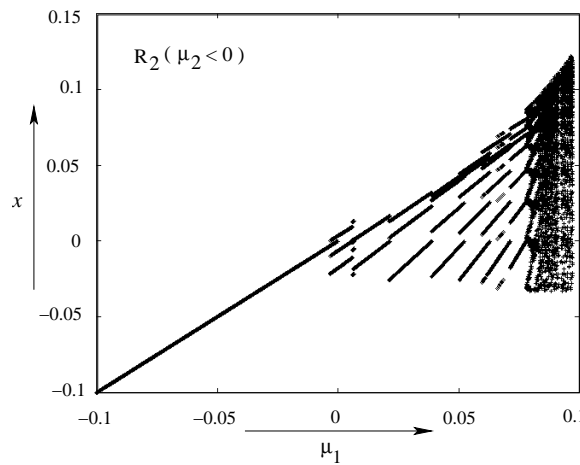
Figure 3. A one-parameter bifurcation diagram with μ_1 as the bifurcation parameter and $\mu_2 = -0.025$. The other parameters $\tau_L = 0.3$, $\delta_L = 0.3$, $\tau_R = 1.5$ and $\delta_R = 0.3$ are in $\mathbf{R}_2$.

In R_2, suppose we keep μ_2 fixed at a negative value, and vary μ_1. Table 1 shows that the period-1 fixed point $\mathbf{X}_L^*$ will exist so long as $\mu_1 < 0$, and the fixed point
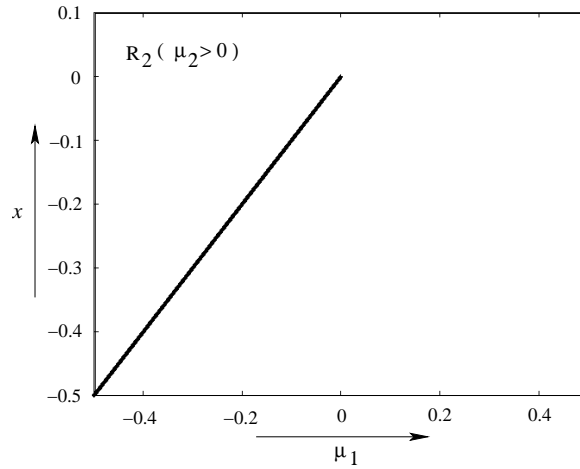
Figure 4. A one-parameter bifurcation diagram with μ_1 as the bifurcation parameter and $\mu_2 = 0.025$. The other parameters $\tau_L = 0.3$, $\delta_L = 0.3$, $\tau_R = 1.5$ and $\delta_R = 0.3$ are in $\mathbf{R}_2$.

$\mathbf{X}_R^*$ will exist for the whole range of μ_1. Now, if we choose the parameters such that $-(1 + \delta_L) < \tau_L < 1 + \delta_L$ and $\tau_R > 1 + \delta_R$, then $\mathbf{X}_L^*$ is stable and $\mathbf{X}_R^*$ is unstable. The representative bifurcation diagram for such a situation is shown in Fig. 3, which shows the period-1 orbit for $\mu_1 < 0$ which disappears for $\mu_2 > 0$. If we take $\mu_2 > 0$ and vary μ_1, then we infer from Table 1 that $\mathbf{X}_L^*$ will exist (and is stable) for $\mu_1 < 0$, but no fixed point will exist for $\mu_1 > 0$. This is what we see in the bifurcation diagram of Fig. 4.
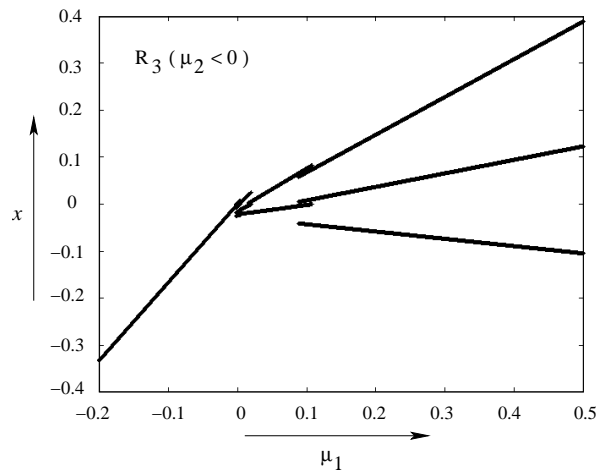

Figure 5. A one-parameter bifurcation diagram with μ_1 as the bifurcation parameter and $\mu_2 = -0.025$. The other parameter $\tau_L = 0.7$, $\delta_L = 0.3$, $\tau_R = 0.3$ and $\delta_R = 0.3$ are in $\mathbf{R}_3$.

In the region R_3, suppose we fix $\mu_2 < 0$ and vary μ_1. Table 1 shows that for $\mu_1 < 0$, $\mathbf{X}_L^*$ will exist but $\mathbf{X}_R^*$ will not. For $\mu_1 > 0$ none of the fixed points will exist. Now, if the parameters are chosen such that $-(1 + \delta_L) < \tau_L < 1 + \delta_L$, then a stable period-1 behavior will be observed so long as $\mu_1 < 0$. This is shown in Fig. 5. On the other hand,
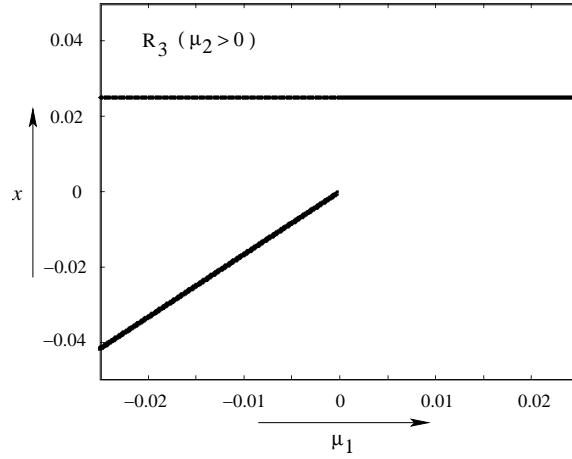
Figure 6. A one-parameter bifurcation diagram with μ_1 as the bifurcation parameter and $\mu_2 = 0.025$. The other parameters $\tau_L = 0.7$, $\delta_L = 0.3$, $\tau_R = 0.3$ and $\delta_R = 0.3$ are in $\mathbf{R}_3$.

if μ_2 is chosen to be positive, for $\mu_1 < 0$ both the fixed points will exist, and for $\mu_1 > 0$ $\mathbf{X}_R^*$ will exist but $\mathbf{X}_L^*$ will not. If the parameters are such that $-(1 + \delta_R) < \tau_R < 1 + \delta_R$, then $\mathbf{X}_R^*$ will occur for the whole parameter range, while $\mathbf{X}_L^*$ will occur only for $\mu_1 < 0$. This is shown in the simulated bifurcation diagram of Fig. 6.
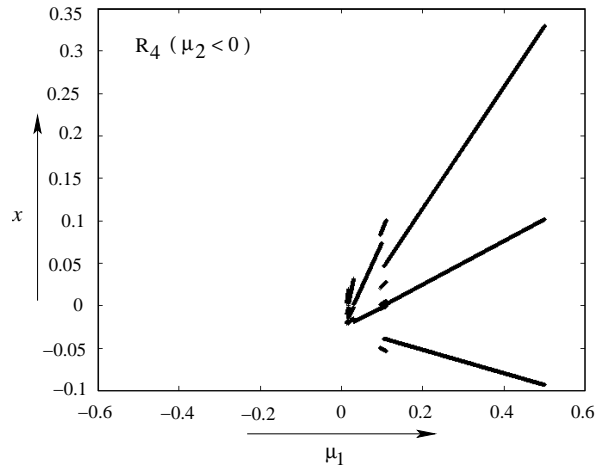

Figure 7. A one-parameter bifurcation diagram with μ_1 as the bifurcation parameter and $\mu_2 = -0.025$. The other parameters $\tau_L = 1.5$, $\delta_L = 0.3$, $\tau_R = 0.3$ and $\delta_R = 0.3$ are in $\mathbf{R}_4$.

Similarly in R_4, suppose we keep μ_2 fixed at a negative value, and vary μ_1. Table 1 shows that the none of the fixed points will exist so long as $\mu_1 < 0$, and for $\mu_1 > 0$ the fixed point $\mathbf{X}_L^*$ will exist (and will be unstable) but $\mathbf{X}_R^*$ will not exist. The representative bifurcation diagram for such a situation is shown in Fig. 7. When $\mu_2 < 0$, the period-1 fixed point is not observed over the whole parameter range, but the existence of the unstable fixed point $\mathbf{X}_L^*$ causes high periodic orbits to occur for $\mu_1 > 0$. On the other hand, if μ_2 is chosen to be positive, for $\mu_1 < 0$ the fixed point $\mathbf{X}_R^*$ will exist but $\mathbf{X}_L^*$
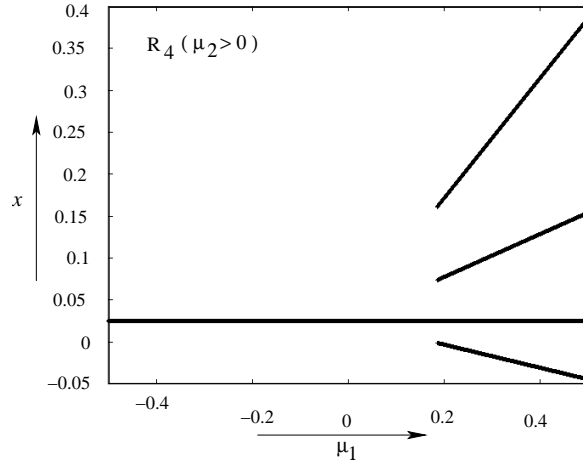
Figure 8. A one-parameter bifurcation diagram with μ_1 as the bifurcation parameter and $\mu_2 = 0.025$. The other parameters $\tau_L = 1.5$, $\delta_L = 0.3$, $\tau_R = 0.3$ and $\delta_R = 0.3$ are in $\mathbf{R}_4$.

will not exist and for $\mu_1 > 0$ both the fixed points will exist. If the parameters are chosen such that $-(1 + \delta_R) < \tau_R < 1 + \delta_R$, then $\mathbf{X}_R^*$ is stable and will occur for the whole parameter range and $\mathbf{X}_L^*$ is unstable. This is shown in the simulated bifurcation diagram of Fig. 8.

4.3. Conditions for existence of period-2 orbits

The conditions for existence of period-2 orbits can be obtained from (25) and (26). First we have to find the values of k_1, k_2 and k_3.

From Section 3, we have

$$\hat{\mathbf{A}}_1 = \mathbf{A}_1 - \frac{1}{p^{**}(-1)} \begin{pmatrix} b_1^* & 0 \\ b_2^* & 0 \end{pmatrix} = \begin{pmatrix} \tau_L - \frac{b_1^*}{p^{**}(-1)} & 1 \\ -\delta_L - \frac{b_2^*}{p^{**}(-1)} & 0 \end{pmatrix},$$

$$\mathbf{B}^* = \begin{pmatrix} b_1^* \\ b_2^* \end{pmatrix} = (-1)^2 (\mathrm{adj}(\mathbf{I} + \mathbf{A}_2))[\mathbf{A}_1 - \mathbf{A}_2]_1$$

$$= \mathrm{adj} \begin{pmatrix} 1 + \tau_R & 1 \\ -\delta_R & 1 \end{pmatrix} \begin{pmatrix} \tau_L - \tau_R \\ -\delta_L + \delta_R \end{pmatrix}$$

$$= \begin{pmatrix} \tau_L - \tau_R + \delta_L - \delta_R \\ \delta_R(1 + \tau_L) - \delta_L(1 + \tau_R) \end{pmatrix},$$

$$d = |\mathbf{I} - \hat{\mathbf{A}}_1| p^{**}(-1) = (1 + \delta_L)(1 + \delta_R) - \tau_L \tau_R$$

and

$$\mathbf{D}^* = (-1)^2 (\mathrm{adj}(\mathbf{I} + \mathbf{A}_2))\mathbf{C}$$

$$= \begin{pmatrix} 1 & -1 \\ \delta_R & 1 + \tau_R \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ \delta_R \end{pmatrix}.$$

Now with the help of above values, we find

$$\hat{\mathbf{K}}_1 = \frac{\text{adj}(\mathbf{I} - \hat{\mathbf{A}}_1)\mathbf{C}}{d} = \frac{1}{d}\begin{pmatrix} 1 \\ -\delta_L - \frac{b_2^*}{p^{**}(-1)} \end{pmatrix},$$

$$\hat{\mathbf{K}}_2 = \frac{\text{adj}(\mathbf{I} - \hat{\mathbf{A}}_1)\mathbf{D}^*}{d} = \frac{1}{d}\begin{pmatrix} 1 + \delta_R \\ -\delta_L - \frac{b_2^*}{p^{**}(-1)} + \delta_R - \delta_R\tau_L + \delta_R\frac{b_1^*}{p^{**}(-1)} \end{pmatrix}.$$

Substituting the parameters, we obtain

$$k_1 = \frac{1}{(1 + \delta_L)(1 + \delta_R) - \tau_L\tau_R}$$

$$k_2 = \frac{1 + \delta_R}{(1 + \delta_L)(1 + \delta_R) - \tau_L\tau_R}$$

and from (27)

$$k_3 = \frac{\tau_L}{(1 + \delta_L)(1 + \delta_R) - \tau_L\tau_R}.$$

Using these expressions, (25) and (26) give

$$\frac{\tau_R\mu_1 + (1 + \delta_R)\mu_2}{(1 + \delta_L)(1 + \delta_R) - \tau_L\tau_R} < 0 \tag{36}$$

$$\frac{(1 + \delta_L)\mu_1 + \tau_L\mu_2}{(1 + \delta_L)(1 + \delta_R) - \tau_L\tau_R} > 0 \tag{37}$$

as the conditions of existence of the period-2 orbit.

We now show that these are indeed the conditions of the existence of the period-2 fixed points in the 2D map.

4.4. Proof of the conditions of existence of period-2 orbits using (28)

Since the system is linear in each side of the border, period two (or a higher period) fixed points cannot exist with all points in L or all points in R. However in some regions of the parameter space, a period two fixed point may exist with one point in L and one point in R. The conditions for existence of period-2 orbit are obtained by solving

$$\begin{pmatrix} m_1^{**} \\ m_2^{**} \end{pmatrix} = \begin{pmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{pmatrix}\begin{pmatrix} m_1^* \\ m_2^* \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\mu_1 \tag{38}$$

$$\begin{pmatrix} m_1^* \\ m_2^* \end{pmatrix} = \begin{pmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{pmatrix}\begin{pmatrix} m_1^{**} \\ m_2^{**} \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\mu_2 \tag{39}$$

where $m_1^* < 0$ and $m_1^{**} > 0$.

Now solving the above two equations we find

$$m_1^* = \frac{\tau_R\mu_1 + (1 + \delta_R)\mu_2}{(1 + \delta_L)(1 + \delta_R) - \tau_L\tau_R} < 0 \tag{40}$$

$$m_1^{**} = \frac{(1+\delta_L)\mu_1 + \tau_L\mu_2}{(1+\delta_L)(1+\delta_R) - \tau_L\tau_R} > 0 \tag{41}$$

These are the same as those obtained from the theory for n-dimensional systems.

This implies that if

$$\tau_L\tau_R > (1+\delta_L)(1+\delta_R)$$

then the period-2 orbit will exist when

$$-\frac{\tau_R}{(1+\delta_R)} < \frac{\mu_2}{\mu_1} < -\frac{(1+\delta_L)}{\tau_L} \tag{42}$$

and if

$$\tau_L\tau_R < (1+\delta_L)(1+\delta_R)$$

then the period-2 orbit will exist when

$$-\frac{\tau_R}{(1+\delta_R)} > \frac{\mu_2}{\mu_1} > -\frac{(1+\delta_L)}{\tau_L}. \tag{43}$$

These two conditions are shown in Fig. 9(a) as the shaded and unshaded regions respectively in the $\tau_L-\tau_R$ space. Now, for particular values of the parameters $\tau_L = 0.7$, $\delta_L = 0.3$, $\tau_R = 0.2$ and $\delta_R = 0.3$ the region of existence of period-2 are depicted in the $\mu_1-\mu_2$ space in Fig. 9(b).
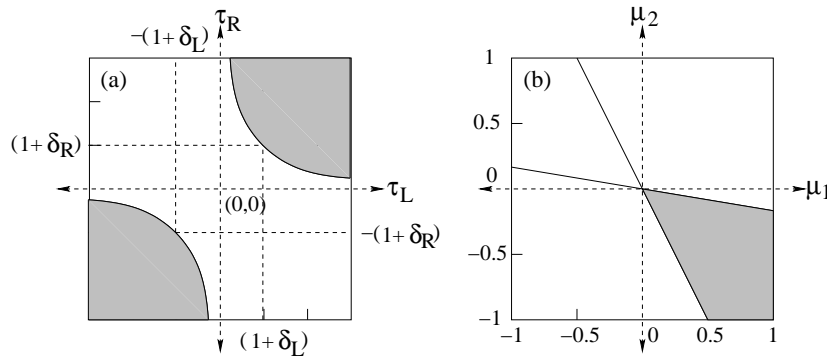

Figure 9. (a) Regions of existence of period-2 points depending on τ_L and τ_R. The period-2 points exist in the shaded region if $\tau_L\tau_R > (1 + \delta_L)(1 + \delta_R)$ and in the unshaded region if $\tau_L\tau_R < (1+\delta_L)(1+\delta_R)$. (b) Regions of existence of period-2 point in $\mu_1-\mu_2$ parameter space. The period-2 points exist in the shaded regions for a set of particularly chosen τ_L, τ_R, δ_L and δ_R.

These equations allow one to calculate the parameter range of existence of the period-2 orbit, when one of the parameters is varied. To illustrate this, we show the bifurcation diagram in Fig. 10, where μ_1 varied and all the other parameters are kept fixed. The stability of the period-2 fixed point is given by the eigenvalues of the matrix $\mathbf{A}_1\mathbf{A}_2$. For the chosen parameter values, these are $\lambda_1 = -0.23 + 0.19i$ and $\lambda_2 = -0.23 - 0.19i$. This implies that if the period-2 fixed point exists, it will be stable. Now from (36) and (37) we find that the period-2 orbit should exist between

$\mu_1 = 0.01346$ and $\mu_1 = 0.1625$. Again, since $\mu_2 < 0$, Table 1 predicts that the period-1 orbit should exist so long as $\mu_1 < 0$. The bifurcation diagram clearly shows that the above predictions are true.
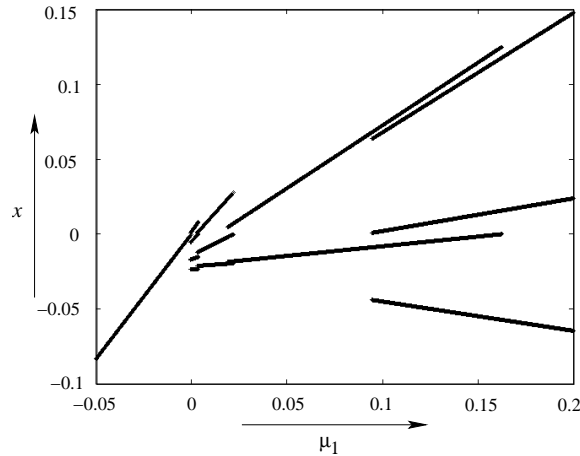

Figure 10. A representative one-parameter bifurcation diagram with μ_1 as the bifurcation parameter. The other parameters are: $\tau_L = 0.7$, $\delta_L = 0.3$, $\tau_R = 0.2$, $\delta_R = 0.3$, $\mu_2 = -0.025$.

5. Conclusions

In view of the fact that a large number of systems of practical interest give rise to discontinuous maps, it has become necessary to develop a comprehensive theory of bifurcations in such systems. In this paper we have taken the first step in that direction by extending Feigin's method that was earlier developed in the context of continuous maps. We have obtained the conditions of existence of the period-1 and period-2 fixed points in general n-dimensional discontinuous maps, and have demonstrated the application of the derived results in a piecewise linear two dimensional discontinuous map.

It may be noted that the results obtained in this paper speak only of the existence of period-1 and period-2 fixed points. In order to obtain information about the occurrence of high-periodic orbits and chaos, one has to consider specific dimension of the system, which cannot be obtained in the general n-dimensional context.

Acknowledgments

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